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A LEAST-SQUARES FINITE ELEMENT METHOD FOR INCOMPRESSIBLE NAVIER–STOKES PROBLEMS

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SUMMARY

A least-squares finite element method based on the velocity–pressure–vorticity formulation was proposed for solving steady incompressible Navier–Stokes problems. This method leads to a minimization problem rather than to the saddle point problem of the classic mixed method and can thus accommodate equal-order interpolations. The method has no parameter to tune. The associated algebraic system is symmetric and positive definite. In order to show the validity of the method for high-Reynolds-number problems, this paper provides numerical results for cavity flow at Reynolds number up to 10 000 and backward-facing step flow at Reynolds number up to 900.

KEY WORDS Finite element Least squares Navier–Stokes equations First-order system Velocity–pressure–vorticity Equal-order interpolations

1. INTRODUCTION

During the past decades various finite element methods for incompressible viscous flows have been developed. Extensive results can be found in the literature.[1–10] Most of these finite element methods are based on the velocity–pressure formulation because of its simpler boundary conditions. Three methods are commonly used to solve the velocity–pressure equations. They are the classic Galerkin mixed method, the penalty method and the streamline upwind Petrov–Galerkin method.

In the classic Galerkin mixed method different elements have to be used to interpolate the velocity and the pressure in order to satisfy the Ladyzhenskaya–Babuška–Brezzi (LBB) condition for the existence of the solution.[11,12] Although for two-dimensional problems quite a few convergent pairs of velocity and pressure elements have been developed, most of these combinations employ basis functions that are not convenient to implement. For three-dimensional problems this difficulty becomes more severe and only rather elaborate constructions can pass the LBB test. Another difficulty is that the matrix associated with the system of linear equations is both non-symmetric owing to the convection terms in the Navier–Stokes equations and non-positive-definite owing to the uncoupled nature of the incompressibility constraint. Therefore direct Gaussian elimination rather than iterative techniques has been considered the only viable method for solving the system. However, for three-dimensional problems the computer resources required by a direct method become prohibitively large.

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The segregated method[13-15] adopts a well-known projection[16] or SIMPLE-type finite difference algorithm.[17] This type of method should be classified as the Galerkin mixed method. The difference between this method and the classic Galerkin mixed method is that the computations of velocity and pressure are decoupled by iteration, and efficiency can be achieved in computer storage and time. Although in many cases equal-order elements were employed, the theoretical justification is still lacking.

In the penalty method (see e.g. References 18 and 19) the pressure is pre-eliminated by penalizing the continuity equation. Involving only velocities, a considerable saving in computing time and computer memory is achieved. However, in many engineering applications the pressure may be the most important design parameter, but the pressure recovered by using the perturbed conservation-of-mass equation exhibits oscillations due to the ill-conditioned pressure matrix. Another disadvantage is the penalty parameter, which for small values causes loss of accuracy and for too large values sometimes prevents convergence to the solution. Furthermore, because of the ill-condition due to the smallness of the parameter, the linear system cannot be solved by iterative techniques. Thus it is difficult to use the penalty method to solve large-scale problems.

The streamline upwind Petrov–Galerkin method[20] has better accuracy and stability than the classic Galerkin formulation. However, it has a parameter which should be tuned. Moreover, the resulting algebraic system is non-symmetric.

In order to overcome these difficulties, we proposed a least-squares finite element method (LSFEM)[21,22] based on the first-order velocity–pressure–vorticity formulation. In this paper we provide more computational results to show the validity of the method for high-Reynolds-number flows. For the sake of completeness we describe the LSFEM for first-order systems in Section 2. Section 3 provides a derivation of the first-order velocity–pressure–vorticity formulation for the Navier–Stokes equations. In Section 4 the performance of the LSFEM is illustrated by the computational results for high-Reynolds-number cavity flows and backward-facing step flows. In Section 5 we discuss the features of the LSFEM. Conclusions are given in Section 6.

2. LSFEM FOR FIRST-ORDER SYSTEMS

The LSFEM of interest here is based on minimizing the residual in differential equations in the L_2-norm. The LSFEM requires that the trial functions be smooth enough to lie in the domain of the differential operator. For example, for second-order differential equations the LSFEM must use C^1-elements, which are not convenient from the computational point of view. In order to use simple C^0-elements, we start from first-order differential equations. Fortunately, for many problems in mechanics and physics the original governing equations, which are derived from the conservation laws and the constitutive laws, are of first order. For historical reasons (convenience for hand calculation and ease of analysis) these equations are converted into higher-order equations with one or few variables. For instance, for incompressible irrotational flows, by introducing the potential or the streamfunction, the first-order incompressibility and irrotationality equations of velocity components are combined into a second-order Laplace equation. After solving the Laplace equation, one has to take differentiation of the potential to obtain the velocity components in which one is really interested. We think that in the computer age this approach is not necessary.

We consider the boundary value problem

$$\mathbf{Lu}=\mathbf{f} \quad \text{in } \Omega, \tag{1}$$

$$\mathbf{Bu}=\mathbf{g} \quad \text{on } \Gamma, \tag{2}$$

where $\mathbf{L}$ is a linear first-order partial differential operator,

$$\mathbf{Lu} = \sum_{i=1}^{n_d} \mathbf{A}_i \frac{\partial \mathbf{u}}{\partial x_i} + \mathbf{Au}, \tag{3}$$

$\Omega \subset \mathbb{R}^{n_d}$ is a bounded domain with a piecewise smooth boundary, Γ, $n_d = 2$ or 3 represents the number of space dimensions, $\mathbf{u}^T = (u_1, u_2, \ldots, u_m)$ is a vector of m unknown functions of $\mathbf{x}$, $\mathbf{A}_i$ and $\mathbf{A}$ are $m \times m$ matrices which depend on $\mathbf{x}$, $\mathbf{f}$ is a given vector-valued function, $\mathbf{B}$ is a boundary algebraic operator and $\mathbf{g}$ is a given vector-valued function on the boundary. Without loss of generality we assume that $\mathbf{g}$ is a zero vector.

Here we do not discuss the existence and uniqueness of the solution to (1), because these depend on the structure and properties of $\mathbf{L}$ and $\mathbf{B}$ and the vector $\mathbf{f}$. In the following discussion it is assumed that problem (1) has a unique solution. We indicate that if there is a solution to (1), then the following least-squares method produces an approximate solution. It does not matter whether the operator $\mathbf{L}$ is elliptic, parabolic or hyperbolic.

Throughout this paper, $L_2(\Omega)$ denotes the space of square-integrable functions defined on Ω with inner product

$$(u, v) = \int_\Omega uv \, d\Omega, \qquad u, v \in L_2(\Omega), \tag{4}$$

and norm

$$\| u \|_0^2 = (u, u), \qquad u \in L_2(\Omega). \tag{5}$$

We define the Sobolev space as

$$H^1(\Omega) = \{ u \in L_2(\Omega) \mid \partial^\alpha u \in L_2(\Omega), \ \forall |\alpha| \leqslant 1 \}, \tag{6}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{n_d}) \in \mathbb{N}^{n_d}$ and $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_{n_d}$, and define the associated norms by

$$\| u \|_1^2 = \sum_{|\alpha| \leqslant 1} \| \partial^\alpha u \|_0^2. \tag{7}$$

For the vector-valued function $\mathbf{u}$ with m components we have the product spaces

$$\mathbf{L}_2(\Omega) = (L_2(\Omega))^m, \tag{8}$$

$$\mathbf{H}^1(\Omega) = (H^1(\Omega))^m \tag{9}$$

and the corresponding norms

$$\| \mathbf{u} \|_0^2 = \sum_{j=1}^m \| u_j \|_0^2, \tag{10}$$

$$\| \mathbf{u} \|_1^2 = \sum_{j=1}^m \| u_j \|_1^2. \tag{11}$$

Considering the boundary condition of the boundary value problem, we also define the function space

$$\mathbf{S} = \{ \mathbf{u} \in (H^1(\Omega))^m \mid \mathbf{Bu} = 0 \text{ on } \Gamma \}. \tag{12}$$

Let us suppose that $\mathbf{f} \in \mathbf{L}_2$ and $\mathbf{L}: \mathbf{S} \to \mathbf{L}_2$. For an arbitrary trial function $\mathbf{u} \in \mathbf{S}$ we define the residual function $\mathbf{R} = \mathbf{Lu} - \mathbf{f}$. The LSFEM is based on minimizing the residual function in a least-squares sense.

We construct the least-squares functional

$$I(\mathbf{u}) = \| \mathbf{Lu} - \mathbf{f} \|_0^2 = (\mathbf{Lu} - \mathbf{f}, \mathbf{Lu} - \mathbf{f}). \tag{13}$$

Taking variation of I with respect to $\mathbf{u}$ and setting $\delta I = 0$ and $\delta \mathbf{u} = \mathbf{w}$ leads to the following least-squares weak statement: find $\mathbf{u} \in \mathbf{S}$ such that

$$(\mathbf{Lw}, \mathbf{Lu}) = (\mathbf{Lw}, \mathbf{f}) \quad \forall \mathbf{w} \in \mathbf{S}. \tag{14}$$

In the approximate analysis we first discretize the domain as a union of finite elements and then introduce an appropriate finite element basis. Let N_e denote the number of nodes for one element and ψ_j denote the element shape functions. If *equal-order interpolations* are employed, i.e. for all unknown variables the same finite element is used, we can write the expansion

$$\mathbf{u}_h(\mathbf{x}) = \sum_{j=1}^{N_e} \psi_j(\mathbf{x}) \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}_j, \tag{15}$$

where $(u_1, u_2, \ldots, u_m)_j$ are the nodal values at the jth node and h denotes the mesh parameter.

Introducing the finite element approximation defined in (15) into the weak statement (14), we have the linear algebraic equations

$$\mathbf{KU} = \mathbf{F}, \tag{16}$$

where $\mathbf{U}$ is the global vector of nodal values. The global matrix $\mathbf{K}$ is assembled from the element matrices

$$\mathbf{K}_e = \int_{\Omega_e} (\mathbf{L}\psi_1, \mathbf{L}\psi_2, \ldots, \mathbf{L}\psi_{N_e})^{\mathrm{T}} (\mathbf{L}\psi_1, \mathbf{L}\psi_2, \ldots, \mathbf{L}\psi_{N_e}) \, \mathrm{d}\Omega, \tag{17}$$

in which $\Omega_e \subset \Omega$ is the domain of the eth element, T denotes the transpose and the vector $\mathbf{F}$ is assembled from the element vectors

$$\mathbf{F}_e = \int_{\Omega_e} (\mathbf{L}\psi_1, \mathbf{L}\psi_2, \ldots, \mathbf{L}\psi_{N_e})^{\mathrm{T}} \mathbf{f} \, \mathrm{d}\Omega, \tag{18}$$

in which

$$\mathbf{L}\psi_j = \psi_{j,x}\mathbf{A}_1 + \psi_{j,y}\mathbf{A}_2 + \psi_{j,z}\mathbf{A}_3 + \psi_j \mathbf{A}. \tag{19}$$

Remark 1

If $\mathbf{L}$ is an elliptic operator, then we have the inequalities

$$(\mathbf{Lw}, \mathbf{Lu}) \leqslant M_0 \| \mathbf{w} \|_1 \| \mathbf{u} \|_1, \qquad (\mathbf{Lu}, \mathbf{Lu}) \geqslant \alpha \| \mathbf{u} \|_1^2 \tag{20}$$

for arbitrary $\mathbf{u}, \mathbf{w} \in S$, where M_0 and α are positive constants. That is, the bilinear form $(\mathbf{Lw}, \mathbf{Lu})$ is continuous and coercive. Following the classic argument (see e.g. Reference 23, pp. 26–28), if the exact solution is smooth enough, we can easily obtain the error estimate

$$\| \mathbf{e} \|_1 = \| \mathbf{u} - \mathbf{u}_h \|_1 \leqslant \frac{M_0 C}{\alpha} h^k, \tag{21}$$

where the constant C is independent of the mesh size h and k denotes the order of complete polynomial of shape functions. Furthermore, the L_2-error estimate can be obtained by the Aubin–Nitsche trick (see e.g. Reference 24, p. 88). These facts indicate that all variables with equal-order interpolations converge at the optimal rate. More general discussion about the mathematical theory of the least-squares finite element method for linear elliptic first-order systems can be found in References 25–27. The incompressible Navier–Stokes problems considered in this paper can be written as a quasi-linear first-order system (see below), which is elliptic when the viscosity is not small. To solve the problems, we have to linearize the system. If the viscosity is large enough, for each linearized step we can expect optimality of the LSFEM. For convection-dominated (hyperbolic) problems the optimality of the LSFEM may not hold in general. However, we can prove that the LSFEM has optimal stability and our numerical experiments show that the LSFEM has near-optimal accuracy. Moreover, the corresponding iteratively reweighted LSFEM, in which each step is the LSFEM, can accurately capture the discontinuity in just one element without any oscillation or diffusion.[28]

3. VELOCITY–PRESSURE–VORTICITY FORMULATION

Consider the following incompressible Navier–Stokes problem: find the velocity $\mathbf{u}=(u_1, u_2, u_3)$ and the pressure p such that

$$\nabla \cdot \mathbf{u}=0 \quad \text{in } \Omega, \tag{22a}$$

$$\mathbf{u} \cdot \nabla \mathbf{u}-\frac{1}{Re}\Delta \mathbf{u}+\nabla p=\mathbf{f} \quad \text{in } \Omega. \tag{22b}$$

Here all variables are non-dimensionalized, $\mathbf{f}=(f_x, f_y, f_z)$ is the body force and Re denotes the Reynolds number, defined as

$$Re=\frac{UL}{v},$$

where L is a reference length, U is a reference velocity and v is the kinematic viscosity. Of course, the boundary conditions should be supplemented to complete the definition of the boundary value problem.

Since the momentum equations (22b) involve second-order derivatives of velocity, direct application of the least-squares method requires the use of inconvenient C^1-elements and produces matrices with large condition number.[26] In order to use the LSFEM described in Section 2, we must consider the governing equations of incompressible flows in the form of first-order systems. There are several ways to do this. For example, one may write the Navier–Stokes equations in the velocity–pressure–stress formulation, which is useful for viscoelasticity and non-Newtonian flow problems, but this formulation has too many variables. Instead, we introduce the vorticity $\boldsymbol{\omega}=\nabla \times \mathbf{u}$ and rewrite the incompressible Navier–Stokes equations in the following first-order quasi-linear velocity–pressure–vorticity formulation:

$$\begin{gathered}\nabla \cdot \mathbf{u}=0, \\ \mathbf{u} \cdot \nabla \mathbf{u}+\frac{1}{Re}\nabla \times \boldsymbol{\omega}+\nabla p=\mathbf{f} \quad \text{in } \Omega, \\ \boldsymbol{\omega}-\nabla \times \mathbf{u}=0.\end{gathered} \tag{23}$$

We shall consider the two-dimensional problem only:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial p}{\partial x}+\frac{1}{Re}\frac{\partial \omega}{\partial y}=f_x,$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{\partial p}{\partial y}-\frac{1}{Re}\frac{\partial \omega}{\partial x}=f_y \quad \text{in } \Omega,$$

$$\omega+\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=0. \tag{24}$$

We can write (24) in the general form of a first-order system (1) in which

$$\mathbf{A}_1=\begin{pmatrix}1 & 0 & 0 & 0\\ u & 0 & 1 & 0\\ 0 & u & 0 & -1/Re\\ 0 & -1 & 0 & 0\end{pmatrix}, \qquad \mathbf{A}_2=\begin{pmatrix}0 & 1 & 0 & 0\\ v & 0 & 0 & 1/Re\\ 0 & v & 1 & 0\\ 1 & 0 & 0 & 0\end{pmatrix},$$

$$\mathbf{A}=\begin{pmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \qquad \mathbf{f}=\begin{pmatrix}0\\ f_x\\ f_y\\ 0\end{pmatrix}, \qquad \mathbf{u}=\begin{pmatrix}u\\ v\\ p\\ \omega\end{pmatrix}. \tag{25}$$

Since the system is of first order, the boundary conditions are very simple and do not involve the derivatives of unknowns. Let $(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5)$ denote the sides of Γ. The unit outward normal vector to Γ is denoted by $\mathbf{n}$ and the tangential vector to Γ by $\mathbf{t}$. We may consider, for instance, the following boundary conditions:

(a) $u=0$, $v=0$ on Γ_1 (the wall)
(b) $u=$constant, $v=0$ on Γ_2 (the far field)
(c) $u=$given, $v=0$ on Γ_3 (the well-developed inflow or outflow)
(d) $u_n=0$, $\omega=0$, $p=$constant on Γ_4 (the free surface)
(e) $u_t=0$, $p=$constant on Γ_5 (the outflow).

We note that in most cases the specification of boundary conditions for the vorticity ω is not necessary. At the solid wall and the well-developed inflow or outflow boundaries we prescribe only the velocity components; no vorticity is involved. At the free surface we specify $\omega=0$.

The quasi-linear problem (25) can be linearized by the successive substitution or Newton's scheme, then the linearized equations can be solved by the LSFEM.

Remark 2

It should be noted that as usual we use Gaussian numerical quadrature to evaluate the element matrices. As explained in our previous paper,[21] the number of 'Gauss' points required for the solution is of some importance. Inspection of the fourth equation in (24) shows that the vorticity ω and the derivatives of velocity components u and v appear simultaneously. When an equal order interpolation is employed, it is impracticable to reduce the residual of this equation to zero throughout. For this reason we must use reduced integration. The necessity of reduced integra-

tion can also be explained from another point of view. The least-squares method with numerical quadrature is equivalent to a weighted collocation least-squares method in which the residual equations are collocated at the interior points in each element, then the algebraic system is approximately solved by the weighted least-squares method. The 'Gauss' points for calculating the element matrices in the least-squares method correspond to the collocation points in the collocation method. The number of collocation points ('Gauss' points) thus cannot be chosen arbitrarily and should be compatible with the number of unknowns. For bilinear elements we use one-point Gaussian quadrature. In this case the total number of unknown nodal values is equal or approximately equal to the total number of residual equations in the collocation method, and the least-squares method is able to force the residuals to be almost zero at the 'Gauss' points; therefore we can expect a good solution at these points. Our numerical experiments reveal that the solution at Gaussian points is very smooth, while the nodal values of p and ω have some oscillation. Therefore we simply average the values at the Gaussian points to obtain the final nodal solution. For triangle elements, even one-point Gaussian quadrature will lead to an overdetermined algebraic system and the least-squares method is not able to force the residuals to be zero. This explains why the use of triangle elements will lead to the locking problem. All the above-mentioned troubles can be completely overcome by using the p-version least-squares method.

4. NUMERICAL RESULTS

The LSFEM described in the previous sections has been tested by solving the driven cavity flow and backward-facing step flow. In this study a simple successive substitution scheme is used to obtain the solution. The velocity field at the previous step is used to calculate the coefficient matrices $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}$. This paper is concerned with the validity of the LSFEM only; thus a Gaussian elimination, which is not efficient, is used to solve the algebraic system. The solutions are updated using an underrelaxation method given as

$$\mathbf{u}^* = \alpha \mathbf{u}^n + (1-\alpha)\mathbf{u}^{n-1}, \tag{26}$$

where α is the underrelaxation number, superscript n denotes the substitution level and superscript * denotes the updated solution. The difference between the results of two consecutive steps is defined as

$$e = \max_{i=1,\ldots,N_n} |u_i^n - u_i^{n-1}|,$$

where i denotes the node and N_n is the total number of nodes. The substitution continues until the difference e becomes less than the tolerance 10^{-4}.

Driven cavity flow

The definition of driven cavity flow is as usual; 50×50 non-uniform bilinear elements are used and the mesh distribution is shown in Figure 1. The smallest element size has $h = 0{\cdot}002$ at the four corners.

The boundary conditions for (u, v) are $u = v = 0$ everywhere except on the top lid where $u = 1$, $v = 0$. We specify $p = 0$ at the centre of the bottom. At the two lower corners the fluid does not flow, i.e. $\partial u/\partial y = \partial v/\partial x = 0$; thus we specify $\omega = 0$. The two upper corners are singular points where $\omega = -\infty$. Since we assume that the velocity components are bilinear functions and $u = 1$, $v = 0$ along the upper boundary, we must have $\omega = -\partial u/\partial y + \partial v/\partial x = -1/h + 0 = -500$. Therefore we specify $\omega = -500$ at the two upper corners as the computational boundary conditions. Our

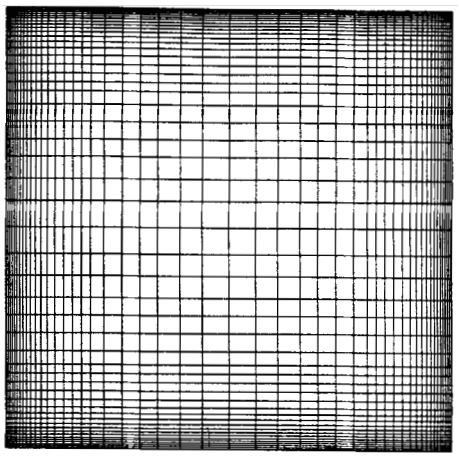
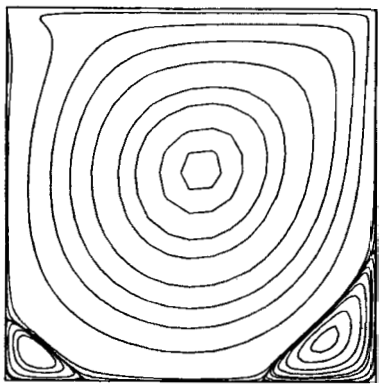
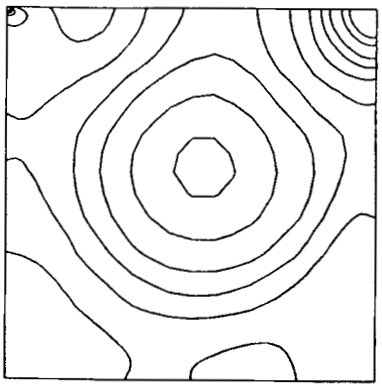


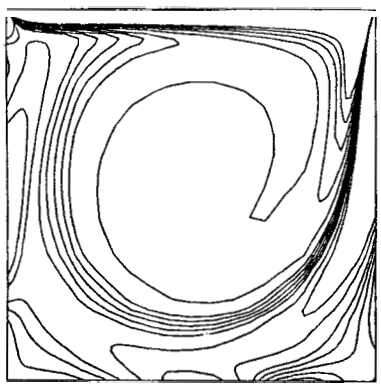
Figure 1. Finite element mesh (50×50 bilinear elements) for cavity flow



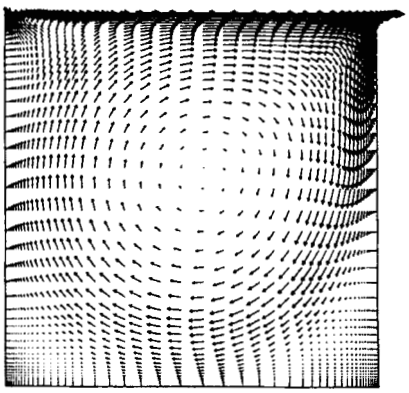
(a) Streamline



(b) Pressure



(c) Vorticity



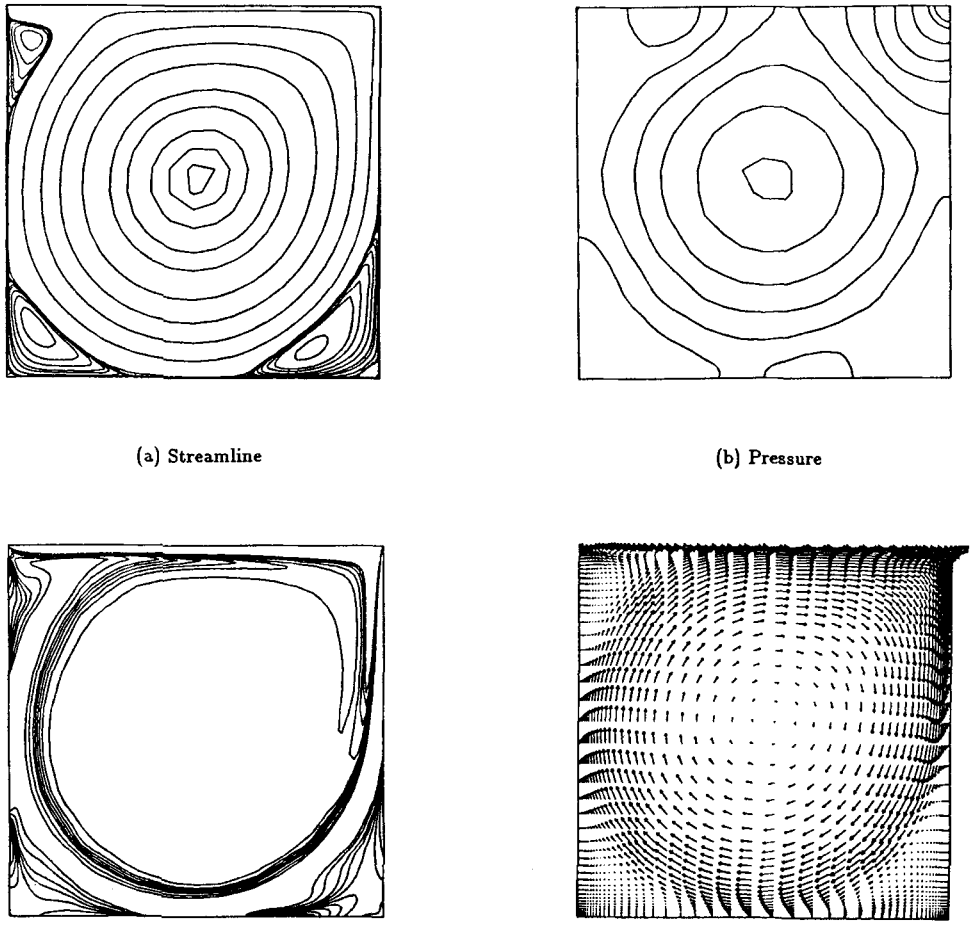
(d) Velocity

Figure 2. Computed results for cavity flow at $Re = 1000$

numerical experiments reveal that these four conditions on the vorticity at the four corners are not necessary, although the vorticity conditions at the two low corners have a little good inference on the solutions around the corners.

The Reynolds numbers considered are 100, 400, 1000, 3200, 5000, 7500 and 10000. In each case $u=v=0$ is taken as an initial guess, i.e. the first step is the solution of the corresponding Stokes problem. No underrelaxation is necessary for $Re<3200$; $\alpha=0{\cdot}8$ is used for $Re\geqslant 3200$. The required numbers of iterations are 8, 12, 14, 22, 28, 40 and 67 respectively.

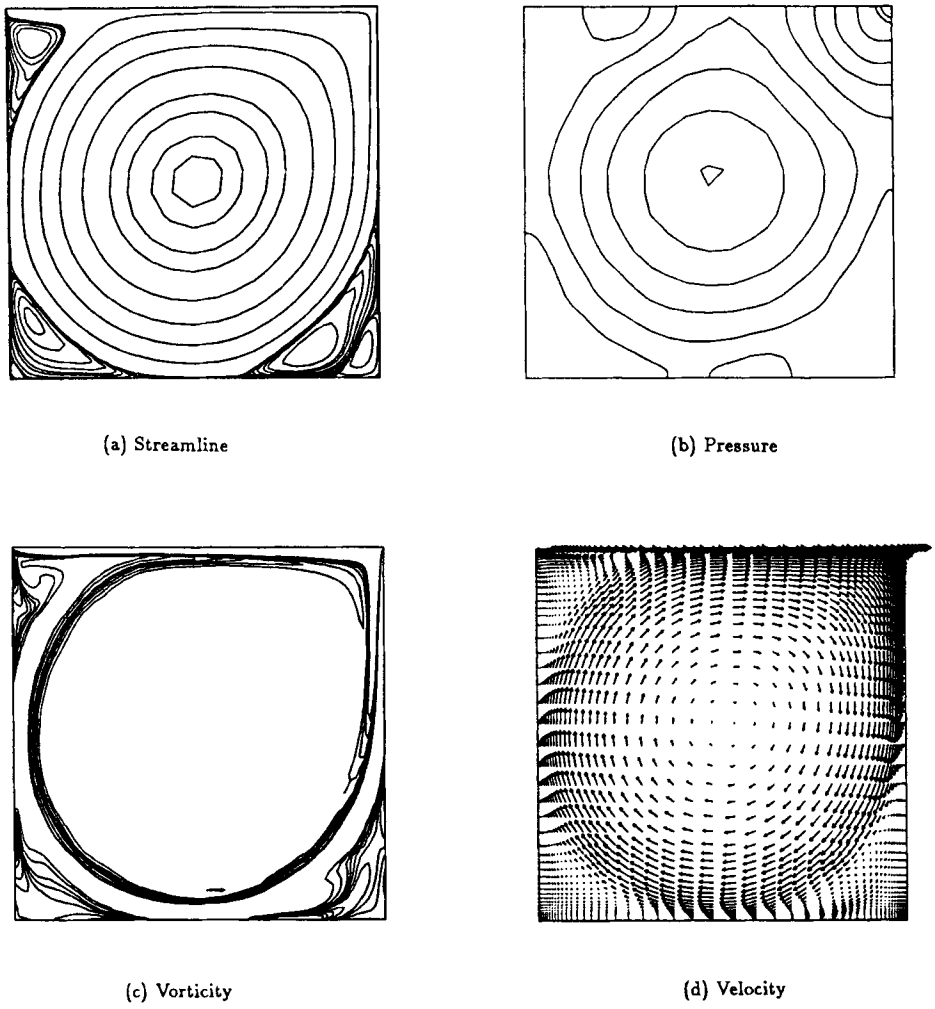
The computed results (streamlines, pressure contours, vorticity contours and velocity vectors) for $Re=1000$, 5000 and 10 000 are shown in Figures 2–4 respectively. Overall, the streamlines and vorticity contours compare rather well with those of Ghia *et al.*[29] except for one region, the lower right eddy at $Re=10\,000$. The size and shape of this small eddy compare more favourably with those of Gresho *et al.*[30] The pressure contours compare well with those of Kim[31] and Sohn *et al.*[32] The horizontal velocity profiles at $x=0{\cdot}5$ compare well with those of Ghia *et al.*[29] in



(a) Streamline (b) Pressure

(c) Vorticity (d) Velocity

Figure 3. Computed results for cavity flow at $Re=5000$



(a) Streamline (b) Pressure

(c) Vorticity (d) Velocity

Figure 4. Computed results for cavity flow at $Re = 10000$

Figure 5. Figure 6 shows the convergence history of the L_2-norm of residuals for $Re = 1000$, 3200 and 5000.

Backward-facing step flow

This example is chosen to compare the computational results with the experimental data of Armaly *et al.*[33] The step has a height of 0·0049 m. The small channel upstream of this step has a height of 0·0052 m. The inlet boundary is located 3·5 step heights upstream of this step. The total length behind the backward-facing step is 45 step heights. A total of 2550 non-uniform bilinear elements (6×15 for the smaller channel and 82×30 for the larger channel) are used, with fine meshes near the step.

The Reynolds number $Re = UD/v$ is based on the hydraulic diameter ($D = 0{\cdot}0104$ m) of the inlet channel and the average velocity ($v = 0{\cdot}6667$ m s^{-1}). The various Reynolds numbers are obtained by varying the kinematic viscosity v.

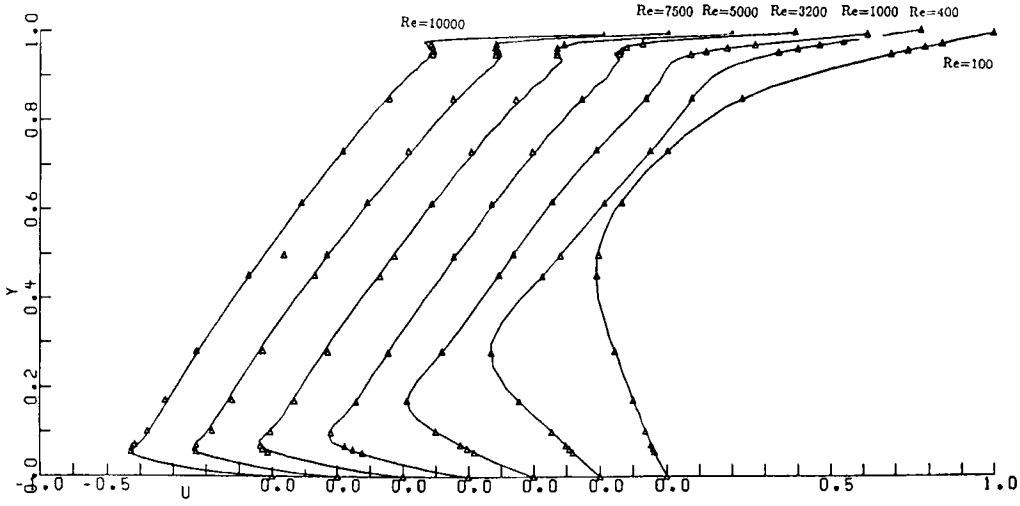


Figure 5. Horizontal velocity profile for cavity flow: ——, present; △, Ghia *et al.*[29]

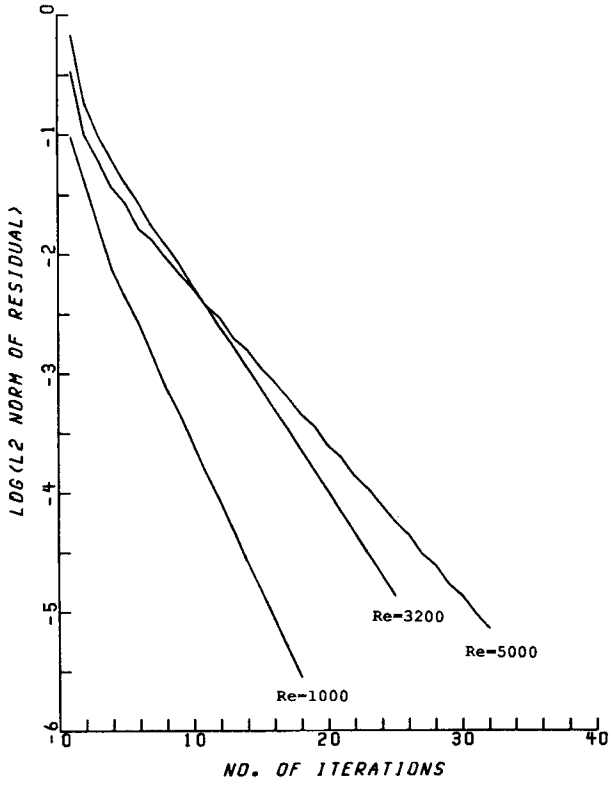


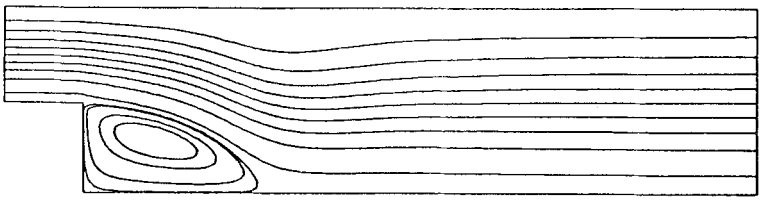
Figure 6. Convergence history for cavity flow

The condition $u=v=0$ is used as an initial guess for $Re=100$, the converged solution for $Re=100$ is used as an initial guess for $Re=200$, and so on. The required numbers of substitutions are 13, 19, 29, 39, 42, 51, 73, 79 and 84 for Reynolds numbers of 100, 200, . . . , 900 respectively. Underrelaxation is not used.

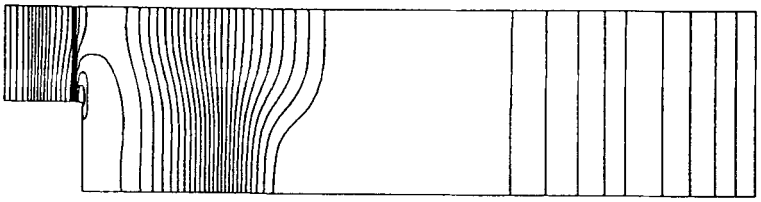
We tested the following combinations of boundary conditions.

(a) A parabolic velocity profile with a centreline velocity of $1{\cdot}0\ \mathrm{m\,s^{-1}}$ and a corresponding linear vorticity profile are imposed at the inlet; $v=0$, $p=0$ are prescribed at the exit boundary; $\omega=0$ is given at the lower step corner.
(b) The velocity profiles are prescribed at the inlet and outlet; $p=0$ is given at the outlet.

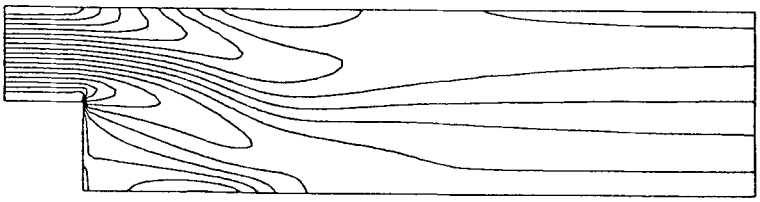
In case (b) no vorticity boundary conditions are involved. For these two cases the numerical results (including the convergence history) have no difference. This confirms that at the boundary, where the velocity components are given, the boundary conditions on the vorticity are not necessary, although the vorticity may be known in advance.



(a) Streamline



(b) Pressure



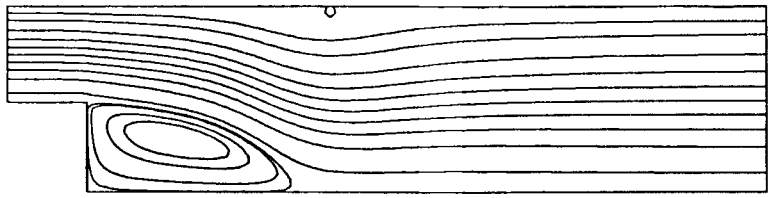
(c) Vorticity

Figure 7. Computed results for backward-facing step flow at $Re=400$

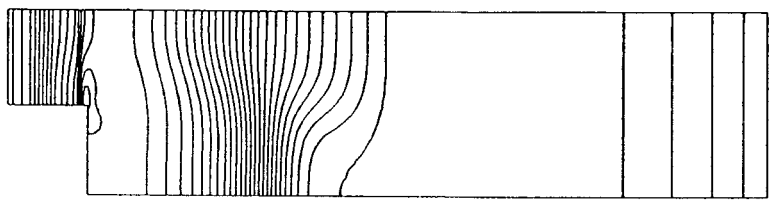
The computed results (streamlines, pressure contours and vorticity contours) for $Re=400$, 500 and 800 are shown in Figures 7–9 respectively. The computed pressure is adjusted by a constant such that $p=0$ at the lower step corner. The reattachment length of the recirculating zone behind the step and the location of detachment and reattachment of another recirculating zone near the upper wall are compared with experimental data in Figure 10, where x_1 is the reattachment location of the primary vortex, x_4 is the separation location of the secondary vortex at the top wall, x_5 is the reattachment location of the secondary vortex and the distance is measured from the expansion step. The reattachment length x_5 compares well with the experimental results. When the Reynolds number is greater than 400, the computed reattachment length x_1 differs from the experimental results, probably owing to the fact that the fluid flow becomes three-dimensional.[33]

5. FEATURES OF THE LSFEM

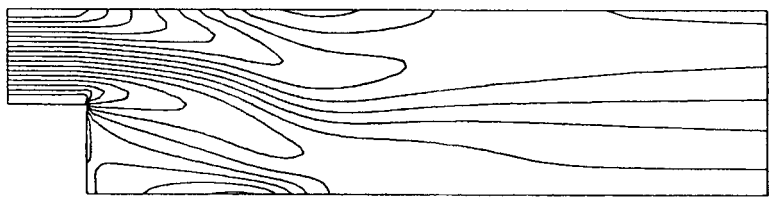
The (u, v, p, ω) formulation (24) is not new. The key issue is how to use it. Johnson[34] formulated the streamline diffusion method for time-dependent incompressible Navier–Stokes problems



(a) Streamline

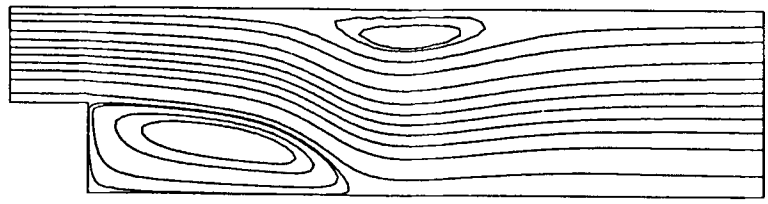


(b) Pressure

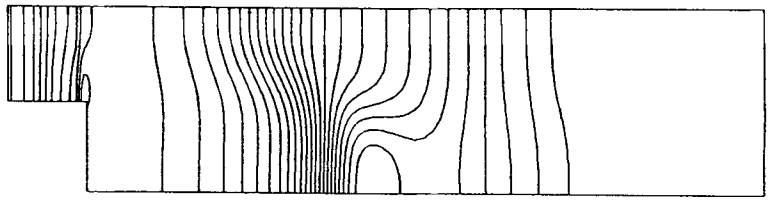


(c) Vorticity

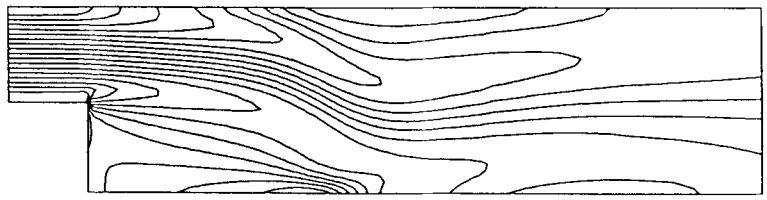
Figure 8. Computed results for backward-facing step flow at $Re=500$



(a) Streamline



(b) Pressure



(c) Vorticity

Figure 9. Computed results for backward-facing step flow at $Re = 800$

based on the (u, v, p, ω) formulation. His method is accurate and stable and can accommodate equal-order interpolations. As with the SUPG method, his method has a parameter which should be tuned and the resulting algebraic system is non-symmetric.

We believe that the LSFEM has more advantages. This method leads to a minimization problem rather than to a saddle point problem, and the choice of combination of elements is thus not subject to the LBB condition. For Stokes problems the numerical experiments exhibit the optimal rate of convergence for all variables with equal-order interpolations.[21] A theoretical error analysis supports this observation.[35] The accommodation of equal-order interpolations is desirable for h-version finite element methods. However, we do not want to emphasize this point too much, because non-equal-order interpolations can be trivially realized in p-version finite element codes. If high-order polynomials are used, e.g. $k \geqslant 10$, it does not matter whether the interpolations are equal-order or non-equal-order.

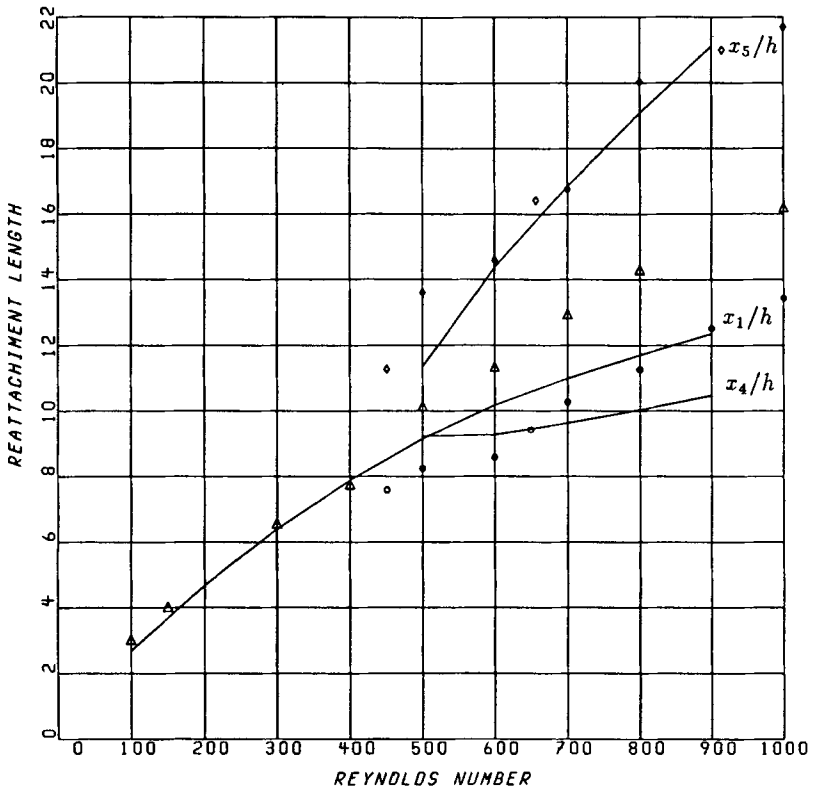


Figure 10. Reattachment length versus Reynolds number for backward-facing step flow: ——, present; △, ○, ◇ experimental—x_1/h, x_4/h, x_5/h respectively

The most important advantage of the LSFEM is that the resulting matrix is symmetric and positive definite. Therefore simple iterative techniques such as the conjugate gradient method can be employed to solve large-scale problems on vector and parallel computers.

Furthermore, there are no added parameters in our LSFEM. We should also emphasize that in our method there is neither artificial dissipation nor upwinding, i.e. the LSFEM is clean and robust.

For the LSFEM the implementation of boundary conditions is quite simple. Although we introduce the vorticity as an independent variable, for most practical problems there are no boundary conditions on the vorticity. Let us explain this by considering the one-dimensional second-order differential equation

$$\frac{d^2u}{dx^2}=f \quad \text{in } (0,1), \qquad u(0)=0, \qquad u(1)=0. \tag{27}$$

where $f\in L_2(0,1)$. By introducing a variable p, it can be rewritten as the first-order system

$$\frac{dp}{dx}=f \quad \text{in } (0,1), \qquad p-\frac{du}{dx}=0 \quad \text{in } (0,1), \qquad u(0)=0, \qquad u(1)=1. \tag{28}$$

In (28) there is no boundary condition on p. It can be proved that the bilinear form of the corresponding LSFEM for (28) without extra BCs is continuous and coercive; therefore the

LSFEM has a unique solution.[21] The same thing is true for high-order multidimensional partial differential equations. In our case, as along as the (u, v, p) formulation subject to boundary conditions on (u, v) has a unique solution, the (u, v, p, ω) formulation of the same problem will have the same solution with no BCs on the vorticity. A similar argument about BCs on the vorticity for the (ω, ψ) formulation can be found in Reference 36 and references cited therein.

6. CONCLUSIONS

A new finite element method for incompressible Navier–Stokes problems is developed. The method is simple, robust and reliable. This method represents a particular application of a unified least-squares finite element method for first-order partial differential equations in computational physics. Further developments are under way for solving 3D problems and time-dependent problems. Theoretical investigation is also needed.

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